S1.1

In steady-state conditions, the output speed does not change. Hence, $J\,d\omega/dt = 0 = T_{em} - T_L - B\omega$. With $B = 0$, the driving torque $T_{em} = T_L = 10$ Nm. With $T_{ref} = T_{em} = K_P\Delta\omega$, the speed error is obtained as $\Delta\omega = \omega^* - \omega = T_L/K_P = 10$ rad/s.

S1.2

With $B = 0$, the open-loop system transfer function $W_S(s) = \omega(s)/\Delta\omega(s)$ is found as K_P/Js. The closed-loop system transfer function is $W_{SS}(s) = \omega(s)/\omega^*(s) = W_S(s)/(1+ W_S(s)) = K_P/(K_P+Js) = 1/(1+sJ/K_P)$. The amplitude $|W_{SS}(j\omega)|$ of the function reduces to 0.7071 (i.e., -3 dB) for $\omega = \omega_{BW} = 2\pi f_{BW} = K_P/J = 100$ rad/s. Hence, the bandwidth frequency is $f_{BW} = 15.91$ Hz.

S1.3

The transfer function $1/(1+sJ/K_P)$ has the numerator $num(s) = 1$ and denominator $den(s) = 0.01s + 1$. The closed-loop step response is obtained by entering the following sequence of commands at the Matlab prompt:

```
>> num = 1;
>> den = [0.01   1];
>> step(num,den)
```

In response, the program prints the figure comprising the step response of the output speed. The rise time is estimated as $\tau_R \approx 20$ ms. The value $f_X = 0.3/\tau_R = 15$ Hz is an acceptable estimate for the closed-loop bandwidth.

S1.4

The open-loop system transfer function $W_S(s) = \omega(s)/\Delta\omega(s)$ is found as $K_P/(Js+B)$. The closed-loop system transfer function is

$$W_{SS}(s)=\frac{\omega(s)}{\omega^*(s)}=\frac{W_S(s)}{1+W_S(s)}=\frac{K_P}{K_P+B+Js}=\left(\frac{K_P}{K_P+B}\right)\left(\frac{1}{1+s\dfrac{J}{K_P+B}}\right).$$

The time constant $\tau = J/(K_P+B)$ determines the response speed. With $\omega^*(s) = \Omega^*/s$, the steady-state value of the output speed is found as

$$\omega(\infty) = \lim_{s \to 0} \left(s\,\omega(s)\right) = \lim_{s \to 0} \left(s \frac{\Omega^*}{s} \frac{K_P}{K_P + B} \frac{1}{1 + s\tau} \right)$$

$$= \frac{K_P}{K_P + B} \Omega^* = 90.909 \frac{\text{rad}}{\text{s}} \; .$$

The speed error equals $\Delta\omega(\infty) = 9.0909$ rad/s.

S1.5

The bandwidth frequency is $f_{BW} = 1/2/\pi/\tau = (K_P+B)/J/2/\pi = 17.5$ Hz.

S2.1

According to Eq. 2.11, the transfer function $W_{LS}(s) = \omega(s)/T_L(s)$ is obtained as $-(1/K_P)/(1+sJ/K_P) = -0.02/(1+0.0002s)$. The closed-loop step response is obtained by entering the following sequence of commands at the Matlab prompt:

```
>> num = -0.02;
>> den = [0.0002   1];
>> step(num,den)
```

Note in the figure obtained that the output speed drops as the load torque increases.

S2.2

The Laplace transform of $T_L(t) = T_{RAMP}\, t\, h(t)$ is obtained as

$$T_L(t) = T_{RAMP} t, \quad T_L(s) = \mathscr{L}\left[T_{RAMP} t\right] = \int_0^{+\infty} T_{RAMP} t\, e^{-st}\, \mathrm{d}t = \frac{T_{RAMP}}{s^2} \qquad .$$

The steady-state value of the output speed is obtained from the final-value theorem:

$$\omega(\infty) = \lim_{s \to 0} \left(s\,\omega(s)\right) = \lim_{s \to 0} \left(s \frac{T_{RAMP}}{s^2} W_{LS}(s) \right)$$

$$= \lim_{s \to 0} \left(\frac{-T_{RAMP}}{s} \frac{1}{K_P + Js} \right) = -\infty \; .$$

Hence, $\Delta\omega(\infty)$ is obtained as $\omega^*(\infty) - \omega(\infty) = +\infty$.

S2.3

From S1.3, the Laplace transform $T_L(s)$ is obtained as T_{RAMP}/s^2. According to Eq. 2.28, the transfer function $W_{LS}(s) = \omega(s)/T_L(s)$ is obtained as $-s/(Js^2+Bs+K_Ps+K_I)$. Therefore,

$$\omega(s) = \frac{T_{RAMP}}{s^2} W_{LS}(s) = \frac{-T_{RAMP}}{s} \frac{1}{Js^2 + (B+K_P)s + K_I} .$$

The steady-state value of the output speed is obtained from the final value theorem:

$$\omega(\infty) = \lim_{s\to 0} (s\,\omega(s))$$

$$= \lim_{s\to 0} \left(s \frac{-T_{RAMP}}{s} \frac{1}{Js^2 + (B+K_P)s + K_I} \right) = \frac{-T_{RAMP}}{K_I} .$$

S2.4

The closed-loop system transfer function $W_{SS}(s)$ is obtained in Eq. 2.27 as

$$W_{SS}(s) = \frac{\omega(s)}{\omega^*(s)}\bigg|_{T_L=0} = \frac{sK_P + K_I}{s^2 J + s(K_P + B) + K_I} .$$

The characteristic polynomial of the system is obtained as

$$d(s) = s^2 + \frac{K_P + B}{J} s + \frac{K_I}{J} = s^2 + 2\xi\omega_n s + \omega_n^2 .$$

The system has one closed-loop zero $z_1 = -K_I/K_P = -3.33$ rad/s and two conjugate complex closed-loop poles $s_{1/2} = -1.55 +/- j\,2.75$. Their natural frequency and damping factor are obtained as

$$\omega_n = \sqrt{\frac{K_I}{J}} = 3.1623 \frac{\text{rad}}{\text{s}}, \quad \xi = \frac{K_P + B}{2J\omega_n} = 0.4902 .$$

S2.5(a)

The closed-loop system transfer function $W_{SS}(s)$ is obtained in Eq. 2.27 as

$$W_{SS}(s) = \frac{\omega(s)}{\omega^*(s)}\bigg|_{T_L=0} = \frac{sK_P + K_I}{s^2 J + s(K_P + B) + K_I} .$$

The characteristic polynomial of the system is obtained as

$$f(s) = s^2 + \frac{K_P + B}{J} s + \frac{K_I}{J} = s^2 + 2\xi\omega_n s + \omega_n^2 .$$

The system has one closed-loop zero z_1 = -K_I/K_P = -3.33 rad/s and two conjugate complex closed-loop poles $s_{1/2}$ = -1.55 +/- j 2.75. Their natural frequency and damping factor are obtained as

$$\omega_n = \sqrt{\frac{K_I}{J}} = 3.1623 \frac{\text{rad}}{\text{s}}, \quad \xi = \frac{K_P + B}{2J\omega_n} = 0.4902 .$$

S2.5(b)

The closed-loop system transfer function $W_{SS}(s)$ is obtained in Eq. 2.41 as

$$W_{SS}(s) = \frac{\omega(s)}{\omega^*(s)}\Big|_{T_L=0} = \frac{K_I}{s^2 J + s(K_P + B) + K_I} = \frac{1}{s^2 \frac{J}{K_I} + s\frac{K_P + B}{K_I} + 1} .$$

It does not have any closed-loop zeros. The characteristic polynomial of the system and the closed-loop poles are the same as in S2.4.

S2.6

The desired traces are obtained by entering the following sequence of commands at the Matlab prompt:

```
>> kp = 0.03;  ki = 0.1;  b = 0.001; j = 0.01;
>> den1 = [ j   (kp+b)  ki];   den2 = den1;
>> num1 = [kp  ki];   num2 = [ki];
>> step( num1,den1,`r`);
>> hold on;
>> step( num2,den2,`b`);
```

The resulting figure, generated by Matlab in response to the *step*() command, comprises transient responses of the output speed obtained with the step change of the speed reference. The red trace is obtained with the proportional action placed in the direct path. It has a larger overshoot than the blue trace, obtained with the proportional gain replaced into the feedback path. In the latter case, the closed-loop system transfer function does not have any zeros.

S2.7

With ξ = 1, the characteristic polynomial assumes the form

$$f(s) = s^2 + \frac{K_P + B}{J} s + \frac{K_I}{J} = s^2 + 2\xi\omega_n s + \omega_n^2 = (s - \omega_n)^2$$

with

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$$\left|W_{SS}\left(j\omega\right)\right| = \left|\frac{\omega(j\omega)}{\omega^*(j\omega)}\right| = \frac{1}{1+\left(\frac{\omega}{\omega_n}\right)^2} = \frac{1}{\sqrt{2}} \quad .$$

The closed loop bandwidth is obtained as

$$\omega_{BW} = \sqrt{\sqrt{2}-1}\ \omega_n \quad .$$

S2.8

The Laplace transform of the speed reference is obtained as

$$\omega^*(s) = \frac{\mathrm{A}^*}{s^2} .$$

The steady-state value of the speed error is obtained from the final value theorem:

$$\Delta\omega(\infty) = \lim_{s\to 0}\left(s\Delta\omega(s)\right) = \lim_{s\to 0}\left(s\left(1-W_{SS}(s)\right)\frac{\mathrm{A}^*}{s^2}\right) = \lim_{s\to 0}\left(sW_E(s)\frac{\mathrm{A}^*}{s^2}\right).$$

With the proportional action placed in the direct path, the steady-state speed error is proportional to the friction B:

$$\lim_{s\to 0}\left(sW_{E1}(s)\frac{\mathrm{A}^*}{s^2}\right) = \mathrm{A}^*\frac{B}{K_I} .$$

With the proportional action replaced into the feedback path, the error becomes

$$\lim_{s\to 0}\left(sW_{E2}(s)\frac{\mathrm{A}^*}{s^2}\right) = \mathrm{A}^*\frac{K_P+B}{K_I} .$$

S2.9

The characteristic polynomial of the closed-loop system is obtained as

$$s^3 + \frac{K_0}{J}s^2 + \frac{K_1}{J}s + \frac{K_2}{J} = (s-\sigma_1)(s-\sigma_2)(s-\sigma_3)$$

where σ_1, σ_2, and σ_3 denote the polynomial zeros. From the above equation,

$$(a) \quad \frac{K_2}{J} = -\sigma_1\sigma_2\sigma_3$$

$$(b) \quad \frac{K_1}{J} = \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3 = -\frac{K_2}{J}\left(\frac{1}{\sigma_1} + \frac{1}{\sigma_2} + \frac{1}{\sigma_3}\right)$$

$$(c) \quad \frac{K_0}{J} = -\sigma_1 - \sigma_2 - \sigma_3 \quad .$$

The characteristic polynomial zeros are, at the same time, the closed-loop system poles. In order to achieve a stable response, all the poles must reside in the left half-plane (Re(σ) < 0).

- In cases with $K_2 = 0$ (a), at least one polynomial zero must be equal to zero.
- With K_1 reduced to zero (b), the sum of the reciprocal values of the zeros is equal to zero. Hence, the requirement that all of the closed-loop poles have a negative component part cannot be met with $K_1 = 0$.
- In cases with $K_0 = 0$ (c), the sum of the closed-loop poles is equal to zero. Therefore, either all of them have their real components equal to zero, or some of them have positive real components.

Starting from a stable parameter set (K_0, K_1, K_2) and reducing any of the gains towards zero brings the closed-loop system to instability. The systems with such properties are found to be conditionally stable. Unconditional stability is the property of closed-loop systems that cannot lose stability by decreasing their feedback gains.

S2.10

The closed-loop characteristic polynomial is obtained by entering the following Matlab commands:

```
>> j = 1; k0 = 1; k1 = 2; k2 = 1;
>> chpol = [1 k0/j  k1/j  k2/j];
```

Zeros of the polynomial are found by

```
>> roots(chpol);
```

and are obtained as $\sigma_{1/2} =$ -0.215 +/- j1.307 and $\sigma_3 = -0.569$.

With $K_{1MIN} = 1$, the stability limit is reached, resulting in the closed-loop system poles $\sigma_{1/2} =$ 0 +/- j and $\sigma_3 = -1$.

S2.11

With K_{2MAX} = 2, the stability limit is reached, resulting in the closed-loop system poles $\sigma_{1/2}$ = 0 +/- j1.41 and σ_3 = -1.

S2.12

The average speed gradually decreases. At the same time, the speed pulsates according to the trapezoidal reference profile. A continuous drop in the output speed is the result of $\Delta T_L \quad T_L - T_{Lff}$ and the lack of a feedback controller.

S2.13

The proportional and integral action of the feedback controller suppresses the effects of ΔT_L. The output speed tracks the speed reference profile, without an error.

S2.14

In the absence of an integral gain, the load torque mismatch produces an offset in the output speed. When both gains are set to zero, the output speed falls continuously, as in S2.12.