S5.1

The transfer function $W_P(s) = 1/s/(Js+B) = numc(s)/denc(s)$ is defined by

```
>> numc = [1];
>> denc = [ 0.01   0.01  0];
>> sysc = tf(numc,denc)
```

The pulse transfer function $W_P(z)$ is obtained as

```
>> Ts = 0.01;
>> sysd = c2d(sysc,Ts, 'zoh')
>> tf(sysd)
>> [numd,dend] = tfdata(sysd, 'v')
```

S5.2

The closed-loop system transfer function is obtained as

$$W_{SS}(s) = \frac{\theta(s)}{\theta^*(s)} = \frac{K_P}{K_P + K_D s + Js^2}$$

$$= \frac{1}{1 + s\frac{K_D}{K_P} + 1 + s^2\frac{J}{K_P}} = \frac{1}{1 + 2s + 1 + s^2}.$$

The step response is obtained by entering the following commands:

```
>> kd = 2; kp = 1; j = 1;
>> num = [1]
>> den = [1 kd/kp   j/kp]
>> step(num,den)
```

From the plot, the rise time is estimated as $\tau_R \approx 3.2$ s, resulting in a closed-loop bandwidth of 0.3/3.2 = 0.0937 Hz.

S5.3

The overshoot of 50% is obtained with $K_D = 0.42$ Nm/(rad/s). With $K_D = 0$, the damping is zero. The amplitude of oscillations in the output position is constant.

S5.4

The transfer function $W_{LS}(s) = \theta(s)/T_L(s)$ is obtained in Eq. 5.16:

$$W_{LS}(s) = \frac{\theta(s)}{T_L(s)}\bigg|_{\theta^*=0} = -\frac{1}{Js^2 + K_D s + K_P} = -\frac{1}{s^2 + 2s + 1}$$

$T_L(t) = T_{LOAD}\, h(t) = 5\, h(t)$ Nm, $T_L(s) = 5/s$. Therefore,

$$\theta(s) = -\frac{5}{s\left(s^2 + 2s + 1\right)}.$$

The inverse Laplace transform $\theta(t) = \mathscr{L}^{-1}(\theta(s))$ is obtained as

```
>> syms s t
>> ilaplace(-5/(s*s+2*s+1)/s)
```

In reply, Matlab provides the response 5*(t+1)*exp(-t)-5. Thereby, $\Delta\theta(\infty) = -\theta(\infty) = 5$ rad. The response of the output position to the step change in the load torque can also be derived by typing the following command sequence:

```
>> num = [ -5];
>> den = [ 1   2   1   0];
>> impulse(num,den)
```

In the resulting figure, the output position approaches -5 rad. In Eq. 5.18, the steady state error in the output position is obtained as:

$$\theta(\infty) = \lim_{s\to 0}\left(s\,\theta(s)\right) = \lim_{s\to 0}\left(s\frac{T_{LOAD}}{s}W_{LS}(s)\right) = -\frac{T_{LOAD}}{K_P}$$

resulting in $\theta(\infty) = -5$ rad.

S5.5

Under the given assumptions, K_{FB} is expressed in [Nm], K_{FB} in [1/rad], and K_P and K_I are without units. The optimized gains are given in Eq. 5.37 and calculated as

$$K_{D\,OPT} = 0.2027\,\frac{2J}{K_{FB}\,K_M\,T^2} = 4054,$$
$$K_{P\,OPT} = 0.03512\,\frac{2J}{K_{FB}\,K_M\,T^2} = 702.4.$$

S5.6

The closed-loop system transfer function numerator and denominator are entered as

```
>> p = 0.03512;  d = 0.2027;
>> num = [p   p  0];  den = [1  (p+d-2)   (1+p)   -d];
```

The step response of the output position is obtained by

```
>> dstep(num,den)
```

The rise time is approximately eight sampling periods. The speed is obtained as the first derivative of the output position:

```
>> stairs(diff(dstep(num,den)))
```

In the absence of the load torque, the driving torque is proportional to the second derivative of the output:

```
>> stairs(diff(diff(dstep(num,den))))
```

S5.7

According to Eq. 5.40,

$$|\omega_1|_{MAX} = fp(|\Delta\theta|) = \sqrt{\frac{2T_{MAX}|\Delta\theta|}{J}} = 447.2 \ \text{rad} \quad .$$

S6.1

The desired transfer function is obtained in Eq. 6.12 as

$$W_{LS}(z) = \frac{\theta(z)}{T^L(z)} = \frac{T^2}{2J} \quad \frac{-(z^2-1)z}{z(z-1)^3 + (z+1)\left[iz^2 + pz(z-1) + d\,(z-1)^2\right]}.$$

The load step response is obtained by entering the following Matlab commands:

```
>> d = 0.2; p = 0.05; i = 0.005; T = 0.001; j = 0.01;
>> num = -T*T/2/j * [1 0 -1 0 ];
>> den = [ 1   (p+d+i-3)   (3+i-d)   (-1-p-d)   d];
>> dstep(num,den)
```

Following the step change in the load torque, the output position sags, and then returns to the reference position within 30–35 sampling intervals. In Chapter 6, the PD position controller gave the steady-state output position error proportional to the load torque.

S6.2

The closed-loop system transfer function is found in Eq. 6.10 as

$$W_{SS}(z) = \frac{(z+1)\, i\, z^2}{z(z-1)^3 + (z+1)\left[iz^2 + pz(z-1) + d\,(z-1)^2\right]} .$$

The closed-loop system transfer function numerator and denominator are entered as

```
>> d = 0.2; p = 0.05; i = 0.005; T = 0.001; j = 0.01;
>> num = [i   i  0   0 ];
>> den = [ 1   (p+d+i-3)   (3+i-d)   (-1-p-d)   d];
```

The step response of the output position is obtained by

```
>> dstep(num,den)
```

The rise time is approximately 12 sampling periods. Compared with the rise time obtained in P5.6/S5.6 with the PD controller, the PID controller rise time is increased by 50%. The speed is obtained as the first derivative of the output position:

```
>> stairs(diff(dstep(num,den)))
```

In the absence of the load torque, the driving torque is proportional to the second derivative of the output:

```
>> stairs(diff(diff(dstep(num,den))))
```

S6.3

The closed-loop zeros in the z-domain are obtained with $roots(num)$:

$z_1 = -1,\ z_2 = 0,\ z_3 = 0$.

The closed-loop poles in the z-domain are are obtained with $roots(den)$:

$p_1 = 0.8351,\ p_2 = 0.7634 + j0.2058,\ p_3 = 0.7634 - j0.2058{,}0,\ p_4 = 0.3831$.

The closed-loop poles in the s-domain are found as

```
>> log(roots(den))/0.001
```

p_1 = -180, p_2 = -234 +j263, p_3 = -234 -j263, p_4 = -959 .

The closed-loop zeros in the s-domain are found as:

```
>> log(roots(num))/0.001
```

$z_1 = +j3141$, $z_2 = s_2 = -\infty$.

S6.4

The closed-loop system transfer function numerator and denominator are entered as

```
>> d = 0.21609; p = 0.0516627; i = 0.0052195; T = 0.001; j = 0.01;
>> num = [i   i  0   0 ];
>> den = [ 1   (p+d+i-3)   (3+i-d)   (-1-p-d)   d];
```

The step response of the output position is obtained by

```
>> dstep(num,den)
```

The rise time is approximately 12 sampling periods. The closed-loop poles are obtained from $roots(den)$, and they are $p_1 \approx p_2 \approx p_3 \approx p_4 \approx 0.6818$.

S6.5

The polynomial $f_{PID}(z)$ has four zeros and only three adjustable gains: p, d, and i.

$$f_{PID}(z) = z^4 - (3 - p - i - d)z^3 + (3 - d + i)z^2 - (1 + p + d)z + d =$$
$$= (z - \sigma_1)(z - \sigma_2)(z - \sigma_3)(z - \sigma_4).$$

Therefore, the four poles cannot be independently set. When three of them are set, the fourth pole is calculated from Eq. 6.15:

$$\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_1\sigma_4 + \sigma_2\sigma_3 + \sigma_2\sigma_4 + \sigma_3\sigma_4$$
$$+ \sigma_1\sigma_2\sigma_3 + \sigma_1\sigma_2\sigma_4 + \sigma_1\sigma_3\sigma_4 + \sigma_2\sigma_3\sigma_4 + \sigma_1\sigma_2\sigma_3\sigma_4 = 7.$$

S6.6

The closed-loop bandwidth is calculated as

$$\omega_{BW} = \omega_p \sqrt{2^{\frac{1}{n}} - 1}\,.$$

The ratio between the bandwidth frequencies obtained with $n = 3$ and $n = 4$ is

$$\frac{\omega_{BW}(n=3)}{\omega_{BW}(n=4)} = \frac{\sqrt{2^{\frac{1}{3}} - 1}}{\sqrt{2^{\frac{1}{4}} - 1}} = 1.172$$

The ratio $f_{BW}^{PD}/\ f_{BW}^{PID}$, obtained in Fig. 6.4, is larger, due to the presence of the closed-loop zeros in the transfer function $W_{SS}(s)$.

S6.7

With tx = 0.015, the driving torque reaches the system limit T_{MAX}, both in acceleration and in braking. The output position reaches the setpoint without an overshoot. For tx = 0.014, an overshoot is observed in the position waveform. With tx = 0.013, the output position exhibits sustained oscillations. A further increase in the reference slope $d\theta^*/dt = 0.02/tx$ brings the system into instability.

S6.8

The analytical considerations are included in Section 7.6.1. The maximum input step is obtained in Eq. 6.27 as:

$$\Delta\theta_{(\max)} = \frac{2T_{MAX}}{J}\left(\frac{K_P\, T}{K_I}\right)^2 \quad .$$